

Notes on Simple Modules over Leavitt Path Algebras

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Abstract

Given an arbitrary graph E and any field K , a new class of simple modules over the Leavitt path algebra $L_K(E)$ is constructed by using vertices that emit infinitely many edges in E . The corresponding annihilating primitive ideals are also described. Using a Boolean subring of idempotents, bounds for the cardinality of the set of distinct isomorphism classes of simple $L_K(E)$ -modules are given. We also append other information about the algebra $L_K(E)$ of a finite graph E over which every simple module is finitely presented.

1 Introduction and Preliminaries

Leavitt path algebras were introduced in [1], [9] as algebraic analogues of graph C^* -algebras and as natural generalizations of Leavitt algebras of type $(1,n)$ built in [17]. The various ring-theoretical properties of these algebras have been actively investigated in a series of papers (see, for e.g., [1], [3], [5], [9], [13], [18], [19], [23]). In contrast, the module theory of Leavitt path algebras $L_K(E)$ of arbitrary directed graphs E over a field K is still at its infancy. The initial organized attempt to study $L_K(E)$ -modules was done in [8] where, for a finite graph E , the simply presented $L_K(E)$ -modules were described in terms of finite dimensional representations of the usual path algebras of the reverse graph \bar{E} of E . As an important step in the study of modules over a Leavitt path algebra $L_K(E)$, the investigation of the simple $L_K(E)$ -modules has recently received some attention (see [10], [11], [12], [14]). Following the ideas of Smith [21], Chen [14] constructed irreducible representations of $L_K(E)$ by using sinks and tail-equivalent classes of infinite paths in the graph E . Chen's construction was expanded in [10] to introduce additional classes of non-isomorphic simple $L_K(E)$ -modules. In section 2 of this paper, we construct a new class of simple left $L_K(E)$ -modules induced by vertices which are infinite emitters and at the same time streamline the process of construction of certain simple modules introduced in [10]. A description of the annihilating primitive ideals of these

simple modules shows that these new simple modules are distinct from (i.e., not isomorphic to) any of the previously constructed simple $L_K(E)$ -modules in [10], [12] and [14]. In section 3, we adapt the ideas of Rosenberg [20] to show that the cardinality of any single isomorphism class of simple left $L_K(E)$ -modules has at most the cardinality of $L_K(E)$. Using a Boolean subring of commuting idempotents induced by the paths in $L_K(E)$, we obtain a lower bound for the cardinality of the set of non-isomorphic simple $L_K(E)$ -modules. In particular, if $L_K(E)$ is a countable dimensional simple algebra, then it will have exactly 1 or at least 2^{\aleph_0} distinct isomorphism classes of simple modules. In section 4, we include some improvements and simplification of the results of [6] dealing with the structure of Leavitt path algebras over which every simple module is finitely presented.

For the general notation, terminology and results in Leavitt path algebras, we refer to [1], [2], [9]. We give below a short outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. All the graphs E that we consider (excepting those studied in section 4) are arbitrary in the sense that no restriction is placed either on the number of vertices in E or on the number of edges emitted by a single vertex. Also K stands for an arbitrary field.

A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular vertex* if it emits a non-empty finite set of edges. An *infinite emitter* is a vertex which emits infinitely many edges. For each $e \in E^1$, we call e^* a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A *path* μ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \dots, n-1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path. Any vertex is considered a path of length 0. The set of all vertices on the path μ is denoted by μ^0 .

A path $\mu = e_1 \dots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be based at the vertex $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

If there is a path from vertex u to a vertex v , we write $u \geq v$. A subset D of vertices is said to be *downward directed* if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is *saturated* if, for any regular vertex v , $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

Given an arbitrary graph E and a field K , the *Leavitt path algebra* $L_K(E)$ is defined to be the K -algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

- (1) $s(e)e = e = er(e)$ for all $e \in E^1$.
- (2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
- (3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if

$e \neq f$.

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

For any vertex v , the *tree* of v is $T(v) = \{w : v \geq w\}$. We say there is a bifurcation at a vertex v , if v emits more than one edge. In a graph E , a vertex v is called a *line point* if there is no bifurcation or a cycle based at any vertex in $T(v)$. Thus, if v is a line point, there will be a single finite or infinite line segment μ starting at v (μ could just be v) and any other path α with $s(\alpha) = v$ will just be an initial sub-segment of μ . It was shown in [12] that v is a line point in E if and only if $vL_K(E)$ (and likewise $L_K(E)v$) is a simple left (right) ideal.

We shall be using the following concepts and results from [23]. A *breaking vertex* of a hereditary saturated subset H is an infinite emitter $w \in E^0 \setminus H$ with the property that $1 \leq |s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty$. The set of all breaking vertices of H is denoted by B_H . For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v, r(e) \notin H} ee^*$. Given a hereditary saturated subset H and a subset $S \subseteq B_H$, (H, S) is called an *admissible pair*. Given an admissible pair (H, S) , the ideal generated by $H \cup \{v^H : v \in S\}$ is denoted by $I_{(H, S)}$. It was shown in [23] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I_{(H, S)}$ for some admissible pair (H, S) . Moreover, $L_K(E)/I_{(H, S)} \cong L_K(E \setminus (H, S))$. Here $E \setminus (H, S)$ is the *Quotient graph* of E in which $(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\}$ and $(E \setminus (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}$ and r, s are extended to $(E \setminus (H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

A useful observation is that every element a of $L_K(E)$ can be written as $a = \sum_{i=1}^n k_i \alpha_i \beta_i^*$, where $k_i \in K$, α_i, β_i are paths in E and n is a suitable integer.

Moreover, $L_K(E) = \oplus_{v \in E^0} L_K(E)v = \oplus_{v \in E^0} vL_K(E)$ (see [1]).

Even though the Leavitt path algebra $L_K(E)$ may not have the multiplicative identity 1, we shall write $L_K(E)(1 - v)$ to denote the set $\{x - xv : x \in L_K(E)\}$. If v is an idempotent or a vertex, we get a direct decomposition $L_K(E) = L_K(E)v \oplus L_K(E)(1 - v)$.

2 A new class of simple modules

Let E be an arbitrary graph. Throughout this section, we shall use the following notation.

For $v \in E^0$ define

$$M(v) = \{w \in E^0 : w \geq v\} \text{ and } H(v) = E^0 \setminus M(v) = \{u \in E^0 : u \not\geq v\}.$$

Clearly $M(v)$ is downward directed. Also, for any vertex v which is a sink or infinite emitter, the set $H(v)$ is a hereditary saturated subset of E . If v is a

finite emitter, it might be that $H(v)$ is not saturated, and that v belongs to the saturation of $H(v)$.

For convenience in writing, we shall denote the Leavitt path algebra $L_K(E)$ by L .

Definition 2.1 *The L -module $S_{v\infty}$:*

Suppose v is an infinite emitter in E . Define $\mathbf{S}_{v\infty}$ to be the K -vector space having as a basis the set $B = \{p : p \text{ a path in } E \text{ with } r(p) = v\}$. Following Chen [14], we define, for each vertex u and each edge e in E , linear transformations P_u, S_e and S_{e^*} on $\mathbf{S}_{v\infty}$ as follows:

For all paths $p \in B$,

$$\begin{aligned} P_u(p) &= \begin{cases} p, & \text{if } u = s(p) \\ 0, & \text{otherwise} \end{cases} \\ S_e(p) &= \begin{cases} ep, & \text{if } r(e) = s(p) \\ 0, & \text{otherwise} \end{cases} \\ S_{e^*}(v) &= 0 \\ S_{e^*}(p) &= \begin{cases} p', & \text{if } p = ep' \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Then it can be checked that the endomorphisms $\{P_u, S_e, S_{e^*} : u \in E^0, e \in E^1\}$ satisfy the defining relations (1) - (4) of the Leavitt path algebra L . This induces an algebra homomorphism ϕ from L to $\text{End}_K(\mathbf{S}_{v\infty})$ mapping u to P_u , e to S_e and e^* to S_{e^*} . Then $\mathbf{S}_{v\infty}$ can be made a left module over L via the homomorphism ϕ . We denote this L -module operation on $\mathbf{S}_{v\infty}$ by \cdot .

Remark 2.2 *The above construction does not work if v is a regular vertex. Specifically, the needed CK-2 relation $\sum_{e \in s^{-1}(v)} S_e S_{e^*} = P_v$ does not hold. Because, on the one hand $(\sum_{e \in s^{-1}(v)} S_e S_{e^*})(v) = 0$ but on the other hand $P_v(v) = v \neq 0$.*

Proposition 2.3 *For each infinite emitter v in E , $\mathbf{S}_{v\infty}$ is a simple left module over $L_K(E)$.*

Proof. Suppose U is non-zero submodule of $\mathbf{S}_{v\infty}$ and let

$$0 \neq a = \sum_{i=1}^n k_i p_i \in U \quad (\#)$$

where $k_i \in K$ and the p_i are paths in E with $r(p_i) = v$ and we assume that the paths p_i are all different.

By induction on n , we wish to show that $v \in U$. Suppose $n = 1$ so that $a = k_1 p_1$. Then $p_1^* \cdot a = k_1 v \in U$ and we are done. Suppose $n > 1$ and assume that $v \in U$ if U contains a non-zero element which is a K -linear combination of less than n paths. Among the paths p_i , assume that p_1 has the smallest length. If p_1 has length 0, that is, if $p_1 = v$ and, for some s , p_s is a path of length > 0 ,

then since $p_s^* \cdot v = 0$, $p_s^* \cdot a \in U$ will be a sum of less than n terms and so by induction $v \in U$. Suppose p_1 has length > 0 . Now $p_1^* \cdot a = a' \in U$. If $p_1^* \cdot a$ is a sum of less than n terms, we are done. Otherwise, $a' = k_1 p_1^* \cdot p_1 + b = k_1 v + b$ where b is a sum of less than n terms with its first non-zero term, say, $k_t p_t$. Then $p_t^* \cdot a' = p_t^* \cdot b \in U$ and $0 \neq p_t^* \cdot b$ is a sum of less than n terms. Hence by induction, $v \in U$ and we conclude that $U = \mathbf{S}_{v\infty}$. ■

The next proposition describes the annihilating primitive ideal of the simple module $\mathbf{S}_{v\infty}$.

Proposition 2.4 *Let v be an infinite emitter. Then*

$$\text{Ann}_{L_K(E)}(\mathbf{S}_{v\infty}) = \begin{cases} I(H(v), B_{H(v)}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| = 0; \\ I(H(v), B_{H(v) \setminus \{v\}}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| \neq 0 \text{ and finite} \\ I(H(v), B_{H(v)}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| \text{ is infinite.} \end{cases}$$

Proof. Let $J = \text{Ann}_{L_K(E)}(\mathbf{S}_{v\infty})$. Clearly $H(v) \subset J$ since for any $u \in H(v)$, $u \not\preceq v$ and so $u \cdot p = 0$ for all p with $r(p) = v$. Indeed $J \cap E^0 = H(v)$.

Suppose $|s^{-1}(v) \cap r^{-1}(M(v))| = 0$ so that $r(s^{-1}(v)) \subseteq H(v)$. Let $u \in B_{H(v)}$. Clearly $u^{H(v)} \cdot p = 0$ if p is a path with $r(p) = v$ and $s(p) \neq u$. On the other hand, if p is a path from u to v with $p = ep'$ where e is an edge, then

$$u^{H(v)} \cdot p = (u - \sum_{f \in s^{-1}(u), r(f) \notin H(v)} f f^*) \cdot p = (e - e)p' = 0.$$

This shows that $I(H(v), B_{H(v)}) \subseteq J$. Now $J \cap E^0 = H(v)$. If J were a non-graded ideal, then it follows from the proof of Theorem 3.12 (iii) of [18], that v will be the base of a cycle c with $c^0 \subset M(v)$. In particular, there is an edge e with $s(e) = v$ and $r(e) \in M(v)$. But this is not possible since $r(s^{-1}(v)) \subseteq H(v)$. Thus J is a graded ideal with $J \cap E^0 = H(v)$. Since $I(H(v), B_{H(v)})$ is the largest graded ideal for which $I(H(v), B_{H(v)}) \cap E^0 = H(v)$, we conclude that $J = I(H(v), B_{H(v)})$.

Suppose $|s^{-1}(v) \cap r^{-1}(M(v))| \neq 0$ and is finite so that $v \in B_{H(v)}$. If $u \in B_{H(v)}$ with $u \neq v$, then the arguments in the preceding paragraph shows that $u^{H(v)} \in J$. But $v^{H(v)} \notin J$ since

$$v^{H(v)} \cdot v = (v - \sum_{f \in s^{-1}(v), L(f) \notin H(v)} f f^*) \cdot v = v \cdot v - 0 = v \neq 0.$$

This shows that $I(H(v), B_{H(v)} \setminus \{v\}) \subseteq J$. Now J is a primitive ideal with $J \cap E^0 = H(v)$. If J were a non-graded ideal, then from the description of the primitive ideals in Theorem 4.3 of [18], we will have $I(H(v), B_{H(v)}) \subseteq J$ and this is not possible since $v^{H(v)} \notin J$. We then conclude that the graded ideal J must be equal to $I(H(v), B_{H(v)} \setminus \{v\})$.

Finally, suppose $|s^{-1}(v) \cap r^{-1}(M(v))|$ is infinite. This means, in particular, there are infinitely many cycles in $M(v)$ based at v . As $M(v) = E^0 \setminus H(v)$, it is then clear from Theorem 3.12 of [18] that J cannot be a non-graded ideal. Also, as $v \notin B_{H(v)}$, the earlier arguments show that $I(H(v), B_{H(v)}) \subseteq J$. Observing that $I(H(v), B_{H(v)}) \cap E^0 = H(v) = J \cap E^0$, we then conclude that $J = I(H(v), B_{H(v)})$. ■

Before proceeding further, we shall review the construction of some of the simple modules introduced in [14] and [10] and refer them as simple modules of type 1, 2 or 3. In this connection, we wish to point out that the notation and terminology used by Chen in [14] is different from those used in papers on Leavitt path algebras such as [4] while we shall follow that of [14].

Type-1 Simple Module: Chen [14] defines an equivalence relation among infinite paths by using the following notation. If $p = e_1 e_2 \cdots e_n \cdots$ is an infinite path where the e_i are edges, then for any positive integer n , let $\tau_{\leq n}(p) = e_1 e_2 \cdots e_n$ and $\tau_{> n}(p) = e_{n+1} e_{n+2} \cdots$. Two infinite paths p and q are said to be *tail equivalent*, in symbols, $p \sim q$, if there exist positive integers m and n such that $\tau_{> n}(p) = \tau_{> m}(q)$. Then \sim is an equivalence relation.

Given an equivalence class of infinite paths $[p]$, let $V_{[p]}$ denote the K -vector space having the set $\{q : q \in [p]\}$ as a basis. Then Chen [14] defines an L -module operation on $V_{[p]}$ making $V_{[p]}$ a left L -module similar to the way the module operation is defined on $\mathbf{S}_{v\infty}$ above, except that the condition that $S_{e^*}(v) = 0$ for any edge e is dropped. Chen [14] shows that the module $V_{[p]}$ becomes a simple L -module.

Type-2 Simple Module: Let w be a sink and \mathbf{N}_w be a K -vector space having as a basis the set $\{p : p \text{ paths in } E \text{ with } L(p) = w\}$. Proceeding as was done above for $\mathbf{S}_{v\infty}$, Chen [14] defines an L -module action on \mathbf{N}_w and shows that \mathbf{N}_w becomes a simple module.

Type-3 Simple Modules: These additional classes of simple L -modules, denoted respectively by $\mathbf{N}_v^{B_{H(v)}}$, $\mathbf{N}_v^{H(v)}$ and $V_{[p]}^f$, were introduced in [10]:

(i) Suppose that v is an infinite emitter such that $v \in B_{H(v)}$. Then we can build the primitive ideal $P = I_{(H(v), B_{H(v)} \setminus \{v\})}$ (see [18]) and the factor ring

$$L_K(E)/P \cong L_K(F)$$

where $F = E \setminus (H(v), B_{H(v)} \setminus \{v\})$. Then $F^0 = (E^0 \setminus H(v)) \cup \{v'\}$,

$$F^1 = \{e \in E^1 : r(e) \notin H(v)\} \cup \{e' : e \in E^1, r(e) = v\}$$

and r and s are extended to F by $s(e') = s(e)$ and $r(e') = v'$ for all $e \in E^1$ with $r(e) = v$. Note that v' is a sink in F and it is easy to see that $M_F(v') = F^0$.

Accordingly, we may consider the Type 2 simple module $\mathbf{N}_{v'}$ of $L_K(F)$ introduced by Chen corresponding to the sink v' in F . Using the quotient map $L_K(E) \rightarrow L_K(F)$, we may view $\mathbf{N}_{v'}$ as a simple module over $L_K(E)$. This simple $L_K(E)$ -module is denoted by $\mathbf{N}_v^{B_{H(v)}}$.

(ii) Suppose v is an infinite emitter and such that $r(s^{-1}(v)) \subseteq H(v)$. Then v is the unique sink in the graph $G = E \setminus (H(v), B_{H(v)})$. Let \mathbf{N}_v be the corresponding Type 2 simple $L_K(E \setminus (H(v), B_{H(v)}))$ -module introduced by Chen. It is clear that \mathbf{N}_v is a faithful simple $L_K(G)$ -module. Consider \mathbf{N}_v as a simple $L_K(E)$ -module through the quotient map $L_K(E) \rightarrow L_K(G)$. This simple module is denoted by $\mathbf{N}_v^{H(v)}$.

(iii) For any infinite path p , $V_{[p]}^f$ is the twisted simple $L_K(E)$ -module obtained from the simple $L_K(E)$ -module $V_{[p]}$. See [10] for details.

Proposition 2.5 *If v is an infinite emitter such that $|s^{-1}(v) \cap r^{-1}(M(v))| \neq 0$ and is finite, then $\mathbf{S}_{v\infty} \cong \mathbf{N}_v^{B_{H(v)}}$.*

Proof. From Proposition 2.4 above and Lemma 3.5 of [10], it is clear that both $\mathbf{S}_{v\infty}$ and $\mathbf{N}_v^{B_{H(v)}}$ are annihilated by the same primitive ideal. Also the K -bases of $\mathbf{S}_{v\infty}$ and $\mathbf{N}_v^{B_{H(v)}}$ are in bijective correspondence. Indeed if $p = p'e$ is a path with $r(p) = v$, then, in the graph F defined in type-3 (i) simple module above, $r(e') = v'$ and $s(e') = s(e) = r(p')$ and so $p'e'$ is a path in F with $r(p'e') = v'$. Then $v \mapsto v'$ and $p = p'e \mapsto p'e'$ is the desired bijection. It is then clear that the map $\phi : \mathbf{S}_{v\infty} \rightarrow \mathbf{N}_v^{B_{H(v)}}$ given by $\phi(v) = v'$ and $\phi(p'e) = p'e'$ extends to an isomorphism from $\mathbf{S}_{v\infty}$ to $\mathbf{N}_v^{B_{H(v)}}$. ■

Proposition 2.6 *If v is an infinite emitter for which $|s^{-1}(v) \cap r^{-1}(M(v))| = 0$, then $\mathbf{S}_{v\infty} \cong \mathbf{N}_v^{H(v)}$.*

Proof. This is immediate after observing that these two simple modules have the same K -basis and the same annihilating primitive ideal. ■

Notation 2.7 *In conformity with the notation used in [10], when v is an infinite emitter for which $|s^{-1}(v) \cap r^{-1}(M(v))|$ is infinite, we shall denote the corresponding simple module $\mathbf{S}_{v\infty}$ by $\mathbf{N}_{v\infty}$.*

Proposition 2.8 *The new simple module $\mathbf{N}_{v\infty}$ is not isomorphic to any of the previously defined simple L -modules of Type 1, 2 or 3.*

Proof. For convenience, we list the simple modules of type 1, 2 and 3 as $\mathbf{N}_w, \mathbf{N}_{v_1}^{B_{H(v_1)}}, \mathbf{N}_{v_2}^{H(v_2)}, V_{[p]}^f, V_{[p]}$. Now $\mathbf{N}_{v\infty} \not\cong V_{[p]}^f$ since the annihilator of $V_{[p]}^f$ is a non-graded primitive ideal ([10], Lemma 2.4) while, as we proved in Proposition 2.4, $\text{Ann}_L(\mathbf{S}_{v\infty}) = I_{(H(v), B_{H(v)})}$ is a graded ideal. The proof that $\mathbf{N}_{v\infty} \not\cong V_{[p]}$ uses the same argument of Chen ([14], Theorem 3.7 (3)). We give the proof for completeness. Suppose $\varphi : \mathbf{N}_{v\infty} \rightarrow V_{[p]}$ is an L -morphism. We claim $\varphi = 0$, that is, $\varphi(v) = 0$. Otherwise, write $\varphi(v) = \sum_{i=1}^n k_i q_i$ where $q_i \in [p]$ and assume that the q_i are all different. Choose n so that $\tau_{\leq n}(q_i)$ are all pairwise different. Now in the definition of $\mathbf{N}_{v\infty}$ as an L -module, $e^* \cdot v = 0$ for all $e \in E^1$ and so $\tau_{\leq n}(q_1)^* \cdot v = 0$, but $\varphi(\tau_{\leq n}(q_1)^* \cdot v) = \tau_{\leq n}(q_1)^* \cdot \varphi(v) = k_1 \tau_{> n}(q_1) \neq 0$, a contradiction. Hence $\mathbf{N}_{v\infty} \not\cong V_{[p]}$.

Since the annihilators of $\mathbf{N}_w, \mathbf{N}_{v_1}^{B_{H(v_1)}}, \mathbf{N}_{v_2}^{H(v_2)}$ and $\mathbf{N}_{v\infty}$ are all graded ideals, it is enough if we can show that the set of vertices belonging to the annihilators of these modules are all different. We first show that $\mathbf{N}_{v\infty} \not\cong \mathbf{N}_w$. Now the vertex set $H(w) \neq H(v)$, since otherwise $M(w) = M(v)$ and this is not possible since $M(w)$ contains a sink (namely, w), while $M(v)$ does not. Hence $\mathbf{N}_{v\infty} \not\cong \mathbf{N}_w$. Likewise, $H(v_2) \neq H(v)$, since otherwise $M(v_2) = M(v)$ which will imply that $v_2 \geq v$ in $M(v_2)$ contradicting the fact that v_2 is a sink in $M(v_2)$. So $\mathbf{N}_{v\infty} \not\cong \mathbf{N}_{v_2}^{H(v_2)}$. Finally, the annihilators of $\mathbf{N}_{v_1}^{B_{H(v_1)}}$ and $\mathbf{N}_{v\infty}$ (being $I_{(H, B_H \setminus \{v_1\})}$ and $I_{(H, B_H)}$ respectively) are different and so $\mathbf{N}_{v_1}^{B_{H(v_1)}} \not\cong \mathbf{N}_{v\infty}$. ■

3 The cardinality of the set of simple $L_K(E)$ -modules

As before, E denotes an arbitrary graph with no restrictions on the cardinality of E^0 or E^1 . We wish to estimate the size of the isomorphism classes of simple left $L_K(E)$ -modules. In this connection, we follow the ideas of Rosenberg [20]. However, we need to modify his arguments for the case of Leavitt path algebras which, among other differences, do not always have multiplicative identities. We first show that, given a fixed simple module S , the cardinality of the set of all maximal left ideals M of $L_K(E)$ such that $L_K(E)/M \cong S$ is at most the cardinality of $L_K(E)$. Using a Boolean subring of idempotents induced by the paths in $L_K(E)$, we obtain a lower bound for the cardinality of the set of non-isomorphic simple $L_K(E)$ -modules. In particular, if $L_K(E)$ is a countable dimensional simple algebra, then it will have either exactly 1 or at least 2^{\aleph_0} distinct isomorphism classes of simple modules.

As before, we shall denote the Leavitt path algebra $L_K(E)$ by L . We begin with a simple description of maximal left ideals of L .

Lemma 3.1 *Suppose M is a maximal left ideal of L . Then for any idempotent $\epsilon \notin M$, $M\epsilon \subset M$ and M can be written as $M = N \oplus L(1 - \epsilon)$ where $N = M \cap L\epsilon = M\epsilon$. Every simple left L -module S is isomorphic to Lv/N for some $v \in E^0$ and some maximal L -submodule N of Lv .*

Proof. Let M be a maximal left ideal of L and $\epsilon = \epsilon^2 \in L \setminus M$. If $x \in M \cap L\epsilon$ then $x = x\epsilon$ and so $M \cap L\epsilon \subset M\epsilon$. By maximality, $M \cap L\epsilon = M\epsilon$, so $M\epsilon \subset M$ for all idempotents ϵ . Writing each $x \in M$ as $x = x\epsilon + (x - x\epsilon)$, we obtain $M = M\epsilon \oplus M(1 - \epsilon) \subset M\epsilon \oplus L(1 - \epsilon)$. By maximality, $M = N \oplus L(1 - \epsilon)$ where $N = M\epsilon = M \cap L\epsilon$ is a maximal L -submodule of $L\epsilon$.

Suppose S is a simple left L -module, say $S = L/M$ for some maximal left ideal of L . Since $L = \bigoplus_{v \in E^0} Lv$ and $M \neq L$, there is a vertex $v \notin M$. By the preceding paragraph, we can write $M = N \oplus L(1 - v)$ where $N = M \cap Lv$. Then $S = [(Lv \oplus L(1 - v))/[N \oplus L(1 - v)]] \cong Lv/N$. ■

Lemma 3.2 *Suppose Lv/N is a simple left L -module with $v \in E^0$. Then, for any vertex u , a simple module Lu/N' is isomorphic to Lv/N if and only if there is an element $a = uav \in Lv$ such that $a \notin N$ and $N'a \subset N$. In this case, $N' = \{y \in Lu : ya \in N\}$.*

Proof. Suppose $\sigma : Lu/N' \rightarrow Lv/N$ is an isomorphism. Let $\sigma(u + N') = x + N$ for some $x \in Lv$. Now $\sigma(u + M) = \sigma(u(u + M)) = u(x + N) = ux + N$. Then $a = ux$ satisfies $a = uav$, $a \notin N$ and $\sigma(u + N') = a + N$. Moreover, $N'a \subset N$ because, for any $y \in N'$, we have $ya + N = y(a + N) = y\sigma(u + M) = \sigma(y + M) = \sigma(0 + M) = 0 + N$. Note that the left ideal $I = \{y \in Lu : ya \in N\}$ contains N' and $I \neq Lu$ since $u \notin I$ (as $ua = a \notin N$). Hence $I = N'$, by the maximality of N' .

Conversely, suppose $N'a \subset N$ for some a satisfying $a \notin N$ and $a = uav$. Define $f : Lu/N' \rightarrow Lv/N$ by $f(y + N') = ya + N$. Now $N'a \subset N$ implies that f is well-defined and is a homomorphism. Now $f \neq 0$ since $f(u + N') = ua + N = a + N \neq N$. As both Lu/N' and Lv/N are simple modules, f is an isomorphism. ■

Lemma 3.3 *Let v be a vertex and $A = Lv/N$ be a simple left L -module. Suppose, for $u, w \in E^0$, $B = Lu/N_1$ and $C = Lw/N_2$ are both isomorphic to A and $b = ubv$ and $c = wcv$ are the corresponding elements satisfying $b, c \notin N$, $N_1b \subset N$ and $N_2c \subset N$ as established in Lemma 3.2. Then $B \neq C$ implies $b \neq c$.*

Proof. Suppose, on the contrary, $b = c$. First of all $u = w$ since otherwise $b = ub = uc = uwc = 0$, a contradiction. Thus $u = w$ and N_1, N_2 are maximal submodules of Lu . Then $N_1b \subset N$ and $N_2b \subset N$ implies $(N_1 + N_2)b = Lub \subset N$. Since $b = ub$, we get $b \in N$, a contradiction. ■

From the preceding Lemmas we get the following Proposition.

Proposition 3.4 (a) *Let Lv/N be a given simple left L -module, where $v \in E^0$. For any fixed vertex u , the cardinality of the set of all maximal submodules N' of Lu (and thus the cardinality of all maximal left ideals of L of the form $M = N' \oplus L(1 - u)$ of L) for which $Lu/N' \cong Lv/N \cong L/M$ is at most the cardinality of uLv .*

(b) *Given a fixed simple left L -module Lv/N , the cardinality of the set of maximal left ideals M of L for which $L/M \cong Lv/N$ is at most the cardinality of L .*

For subsequent applications, we obtain a sharpened version of Lemma 3.2 as follows.

Lemma 3.5 *Let $v \in E^0$ and Lv/N be a simple left L -module. Then, the maximal left ideals M of L for which $L/M \cong Lv/N$ are precisely the annihilators in L of non-zero elements $a + N$ of Lv/N with $a = uav$ for some vertex u .*

Proof. Suppose $L/M \cong Lv/N$ for some maximal left ideal M of L . By Lemma 3.1, we can write $M = N' \oplus L(1 - u)$ where u is a vertex, $u \notin M$ and $N' = M \cap Lu$. By Lemma 3.2, there is an element $a = uav \notin N$ so that $a + N$ is non-zero and $N' = \{y \in Lu : ya \in N\} = \{y \in Lu : y(a + N) = N\}$. It is then clear that $M = \{L \in L : L(a + N) = N\}$.

Conversely, suppose the left ideal I is the annihilator in L of some non-zero element $a + N$ of Lv/N , where $a \in uLv$ for some vertex u . Let $N' = \{y \in Lu : y(a + N) = N\} = \{y \in Lu : ya \in N\}$. Now $N' \neq Lu$ since $u \notin N'$ due to the fact that $ua = a \notin N$. Define $\phi : Lu/N' \rightarrow Lv/N$ by $\phi(ru + N') = rua + N$. Clearly ϕ is a well-defined homomorphism and $\phi \neq 0$, as $\phi(u) = ua + N = a + N$. If $rua + N = N$, then $ru(a + N) = N$, so $ru \in N'$ and $ru + N' = N'$. Thus $\ker(\phi) = 0$. Since Lv/N is simple, $\phi : Lu/N' \rightarrow Lv/N$ is an isomorphism. In particular, N' is a maximal L -submodule of Lu . Then $M = N' \oplus L(1 - u)$ is a

maximal left ideal of L , $L/M \cong Lv/N$ and $M \subset I$. By maximality, $M = I$, the annihilator of $a + N$ in L . ■

In the context of Proposition 3.4(b), our next goal is to investigate the size of the set of all non-isomorphic simple left L -modules. Towards this end, we consider maximal left ideals of L that arise from a specified Boolean subring of idempotents in L .

A special Boolean subring B of L : Let $S = E^0 \cup \{\alpha\alpha^* : \alpha \text{ a finite path in } E\} \cup \{0\}$. Observe that elements of S are commuting idempotents. Moreover, if $a, b \in S$, then it is easy to see that $ab \in S$. Let B be the additive subgroup of L generated by S . Define, for any two elements $a, b \in S$, $a \triangle b = a + b - 2ab$ and $a \cdot b = ab$. Then B becomes a Boolean ring under the operations \triangle and \cdot .

Define a partial order \leq on B by setting, for any two elements $a, b \in B$, $a \leq b$ if $ab = a$. Then B becomes a lattice under the operations, $a \vee b = a + b - ab$ and $a \wedge b = ab$.

Proposition 3.6 (a) *If M' is a maximal left ideal of L , then $M = M' \cap B$ is a maximal ideal of B and $M' = N \oplus L(1 - v)$ for some vertex $v \notin M'$ where $N = M' \cap Lv = M'v$ and $M = Mv \oplus B(1 - v)$.*

(b) *Every maximal ideal M of B embeds in a maximal left ideal P_M of L such that $P_M \cap B = M$. Thus different maximal ideals M_1, M_2 of B give rise to different maximal left ideals P_{M_1}, P_{M_2} .*

Proof. (a) If M' is a maximal left ideal of L , then clearly $M = M' \cap B$ is an ideal of B . To show that M is maximal, it is enough if we show that M is a prime ideal of B . Suppose $x, y \in B$ such that $xy \in M$ and $x \notin M$. Since $Lx + M' = L$, we can write $y = rx + m'$ where $r \in L$ and $m' \in M'$. Then $y = y^2 = rxy + m'y$. By Lemma 3.1, $m'y \in M'y \subset M'$ and so $y \in M' \cap B = M$. Thus M is a maximal ideal of B . Let v be a vertex with $v \notin M'$. By Lemma 3.1, $M' = N \oplus L(1 - v)$ where $N = M'v$. Note that $v \in B$ and $Mv \subset M$, as M is an ideal. Thus $M = Mv \oplus M(1 - v) \subset Mv \oplus B(1 - v)$. By maximality, $M = Mv \oplus B(1 - v)$.

(b) Let M be a maximal ideal of B . Then there is at least one vertex $v \notin M$. Because if $E^0 \subset M$, then for every path α with, say $s(\alpha) = u$, $\alpha\alpha^* = u\alpha\alpha^* \in M$, as M is an ideal of B . This implies $M = B$, a contradiction. We now claim that the left ideal $LMv \neq Lv$. Suppose, by way of contradiction, assume that $v \in LMv$, so that $v = \sum_{i=1}^k r_i m_i v$ where $m_i \in M$ and $r_i \in L$. Observing that $m = m_1 \vee \dots \vee m_k$ belongs to the ideal M and satisfies $m_i m = m_i$ for all i , we get $mv = vm = \sum_{i=1}^k r_i m_i mv = \sum_{i=1}^k r_i m_i v = v$. This is not possible, since $mv \in M$ while $v \notin M$. Thus LMv is a proper L -submodule of Lv and hence can be embedded in a maximal L -submodule N of Lv . Writing each element $x \in M$ as $x = xv + (x - xv)$ we see that M embeds in the maximal left ideal $P_M = N \oplus L(1 - v)$. By the maximality of M , it is clear that $P_M \cap B = M$. This implies that if $M_1 \neq M_2$ are maximal ideals of B embedding, as above, in maximal left ideals P_{M_1} and P_{M_2} of L , then $P_{M_1} \neq P_{M_2}$. ■

Corollary 3.7 *The cardinality of the set of all maximal left ideals of L is at least the cardinality of the set of all maximal ideals of B .*

For each maximal ideal M of B , choose one maximal left ideal $P_M = N \oplus L(1 - v)$ where $N = P_M v$ as constructed in Proposition 3.6(b). Let \mathbf{T} denote the set of all such maximal left ideals P_M of L . We shall call such P_M a *Boolean maximal left ideal corresponding to the maximal ideal M of B* and call the simple module L/P_M a Boolean simple module.

From Proposition 3.6 it is clear that for each vertex v there is a Boolean maximal left ideal $P_M = P_M v \oplus L(1 - v)$ not containing v . Because, given v we can find a maximal left ideal Q of L not containing v . Clearly $Q \cap B = M$ is a maximal ideal in B not containing v . Then proceed as on Proposition 3.6(b), to construct the Boolean maximal left ideal P_M corresponding to M and, as noted there, $P_M = P_M v \oplus L(1 - v)$.

Proposition 3.8 *Let Lv/N be a fixed simple left L -module where $v \in E^0$ and N is a maximal L -submodule of Lv . Let $\mathbf{S}_{v,N} = \{P_M \in \mathbf{T} : L/P_M \cong Lv/N\}$. Let $\sigma = |\mathbf{S}_{v,N}|$ and write $\mathbf{S}_{v,N} = \{P_{M_\alpha} = P_{M_\alpha} v_\alpha \oplus L(1 - v_\alpha) : v_\alpha \in E^0, \alpha < \sigma\}$. Then*

(a) $|\mathbf{S}_{v,N}| \leq \dim_K(Lv/N)$;

(b) *the cardinality of the set of all maximal left ideals P of L such that $L/P \cong Lv/N$ is $\leq \sum_{\alpha < \sigma} |v_\alpha Lv_\alpha|$.*

Proof. (a) By Lemma 3.5, each $P_{M_j} \in \mathbf{S}_{v,N}$ annihilates an element $x_j = a_j + N \in Lv/N$. Regarding Lv/N as a K -vector space, we claim that these elements x_j (corresponding to the various $P_{M_j} \in \mathbf{S}_{v,N}$) must be K -independent. To justify this, suppose a finite subset of the elements x_j , with $j = 1, \dots, n+1$, satisfy

$$\sum_{j=1}^{n+1} k_j x_j = 0 \dots \dots \dots (*)$$

where, for each j , $k_j \in K$ and the maximal ideal P_{M_j} is the corresponding annihilator of the element x_j . Observe that the maximal ideals of B satisfy the Chinese remainder theorem and so, corresponding to the finite set M_1, \dots, M_n, M_{n+1} of maximal ideals of B , there is an element $b \in \cap_{i=1}^n M_i$ such that $b \notin M_{n+1}$ so that the ideal generated by $\cap_{i=1}^n M_i$ and M_{n+1} is B . Since the vertex set $E^0 \subset B$, we then see that $(\cap_{j=1}^n P_{M_j}) + P_{M_{n+1}} = L$ and so there is an element $a \in \cap_{j=1}^n P_{M_j}$, but $a \notin P_{M_{n+1}}$. Since a annihilates x_1, \dots, x_n , multiplying the equation (*) on the left by the element a , we get $k_{n+1} a x_{n+1} = 0$ which implies $k_{n+1} = 0$. Proceeding like this, we establish the independence of the elements x_j . Thus the elements x_j can be regarded as part of a basis of Lv/N . Since distinct maximal left ideals P_{M_j} correspond to different such elements x_j in a basis of Lv/N (Lemma 3.3), we conclude that $|\mathbf{S}_{v,N}| \leq \dim_K(Lv/N)$. ■

(b) Now, for a fixed α , Proposition 3.4(a) implies that the cardinality of the set of all the maximal left ideals P with $P \cap B = P_{M_\alpha} \cap B$ (so $P = P v_\alpha \oplus L(1 - v_\alpha)$) and satisfying $L/P \cong L/P_{M_\alpha} (\cong Lv/N)$ is $\leq |v_\alpha Lv_\alpha|$. So the cardinality of the set of all maximal left ideals P such that L/P is isomorphic to Lv/N is $\leq \sum_{\alpha < \sigma} |v_\alpha Lv_\alpha|$.

Lemma 3.9 *If $\text{Soc}(L) = 0$ and the graph E satisfies Condition (L), then the Boolean ring B is atomless, that is, it has no minimal elements.*

Proof. Since $\text{Soc}(L) = 0$, the graph E cannot have any line points and, in particular, has no sinks. Suppose, by way of contradiction, B has a minimal element m so that, for all $b \in B$, either $mb = 0$ or $mb = m$. In order to reach a contradiction, we first claim that m can be taken to be a monomial of the form $\gamma\gamma^*$ for some path γ . To see this, if $m = v$ is a vertex, then as v is not a sink, it will be the source of some path α and in that case $m = m\alpha\alpha^* = v\alpha\alpha^* = \alpha\alpha^*$. Likewise, suppose $m = \sum_{i=1}^k t_i \alpha_i \alpha_i^*$ where $t_i \in K$ and $\alpha_i \alpha_i^* \neq \alpha_j \alpha_j^*$ for all i, j . Assume, without loss of generality that α_1 is of maximal length. Then, observing that $\alpha_1 \alpha_1^* \alpha_i \alpha_i^* \neq 0$ implies that $\alpha_1 \alpha_1^* \alpha_i \alpha_i^* = \alpha_1 \alpha_1^*$, we conclude that $m = \alpha_1 \alpha_1^* m = \sum_{j=1}^r \alpha_1 \alpha_1^* = r \alpha_1 \alpha_1^*$. Since $m^2 = m$, $r = 1$ and we conclude that $m = \alpha_1 \alpha_1^*$. We thus conclude that $m = \gamma\gamma^*$ for some path γ with $s(\gamma) = u$. Suppose $r(\gamma) = w$ (w may be equal to u). Since w cannot be a line point, there is a vertex in $T(w)$ which is either a bifurcation vertex or is the base of a cycle in $T(w)$. Since every cycle has an exit (due to Condition (L)), $T(w)$ will always contain a bifurcation vertex w' with $w' = s(e) = s(f)$ for some edges $e \neq f$. Denoting a path from w to w' by δ , we obtain, by the minimality of $m = \gamma\gamma^*$,

$$\gamma\gamma^* = \gamma\gamma^* \gamma \delta e e^* \delta^* \gamma^*.$$

Multiplying on the right by $\gamma\delta f$, we get $\gamma\gamma^* \gamma \delta f = \gamma\gamma^* \gamma \delta e e^* \delta^* \gamma^* \gamma \delta f$ and from this we get $\gamma\delta f = \gamma\gamma^* \gamma \delta e e^* f = 0$, a contradiction. This proves that the Boolean ring B has no minimal element. ■

Theorem 3.10 *Let E be an arbitrary graph satisfying Condition (L). If $L = L_K(E)$ is a countable dimensional K -algebra with $\text{Soc}(L) = 0$, then L has at least 2^{\aleph_0} distinct isomorphism classes of simple left L -modules.*

Proof. Consider the Boolean ring B of L defined earlier. By Lemma 3.9, the Boolean ring B has no minimal elements. Thus B is a countable atomless Boolean ring without identity. In this case, it is well-known (see [16] or Theorems 1, 8 and 13 in [22]) that the space X of all maximal ideals of B is a locally compact totally disconnected Hausdorff space with no isolated points. Let X^* be the one-point compactification of X obtained by the adjunction of a single non-isolated point to X . Now X^* is homeomorphic to the Cantor set (see [22]) and so X^* , and hence X , has cardinality 2^{\aleph_0} . Thus B has 2^{\aleph_0} distinct maximal ideals. From Corollary 3.7 we conclude that there is a set \mathbf{T} of 2^{\aleph_0} distinct Boolean maximal left ideals of L obtained from the ideals of B . Now for any given maximal ideal $P = N \oplus L(1 - v) \in \mathbf{T}$, the set $S_P = \{Q \in \mathbf{T} : L/Q \cong L/P\}$ is countable since L and hence Lv/N has countable K -dimension and, by Proposition 3.8, $|S_P| \leq \dim_K(Lv/N)$. Since $|\mathbf{T}| = 2^{\aleph_0}$, L has 2^{\aleph_0} non-isomorphic Boolean simple L -modules of the form L/P where $P \in \mathbf{T}$. Consequently, L has at least 2^{\aleph_0} non-isomorphic simple left L -modules. ■

Corollary 3.11 *Let E be an arbitrary graph. If $L = L_K(E)$ is a simple countable dimensional K -algebra, then L has either exactly 1 or at least 2^{\aleph_0} distinct isomorphism classes of simple left L -modules.*

Proof. If $\text{Soc}(L) \neq 0$, then, by simplicity, $L = \text{Soc}(L)$ and all the simple left L -modules are isomorphic. Suppose $\text{Soc}(L) = 0$, then the simplicity of L implies that the graph E satisfies Condition (L) (see [1]). We then obtain the desired conclusion from Theorem 3.10. ■

Remark 3.12 *The method of proof of Theorem 3.10 breaks down if E is an uncountable graph. Because, unlike the case of a countable atomless Boolean ring, the set of maximal ideals of an uncountable atomless Boolean ring B may not have the desired larger cardinality than $|B|$, unless some conditions such as completeness of B holds, or if B has an independent subset of cardinality $|B|$ (I am grateful to Professor Stefan Geschke for this remark. See [16] for details). When E is an uncountable graph, the Boolean ring B that we constructed in the proof of Theorem 3.10 need not be complete and also need not have a large enough independent subset.*

4 Finitely presented simple modules

Let E be a finite graph. It was shown in [10] that every simple left $L_K(E)$ -module is finitely presented if and only if distinct cycles in E are disjoint, that is, they have no common vertex. Interestingly, in [5], this same condition on the graph E is shown to be equivalent to the algebra $L_K(E)$ having finite Gelfand-Kirillov dimension. Further, Theorem 1 of [6] shows that if the graph E has the stated property, then $L = L_K(E)$ is the union of a finite ascending chain of ideals

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L$$

where I_0 is a direct sum of finitely many matrix rings $M_n(K)$ over K with $n \in \mathbb{N} \cup \{\infty\}$ and, for $j \geq 1$, each successive quotient I_j/I_{j-1} is a direct sum of finitely many matrix rings $M_n(K[x, x^{-1}])$ over $K[x, x^{-1}]$ with $n \in \mathbb{N} \cup \{\infty\}$. In this section, we show that the converse of the above statement holds and obtain an improved version of the statement and proof of Theorem 1 of [6] (see Theorem 4.2 below). An easy proof of Theorem 2 of [6] is also pointed out.

We begin with the following easily derivable Lemma which was implicit in [1] and was proved in [15].

Lemma 4.1 *Let E be any graph and let H be a hereditary subset of vertices in E . If w is the base of a closed path and if $w \in \bar{H}$ the saturated closure of H , then $w \in H$.*

In addition to proving the converse of Theorem 1 of [6], the next theorem consolidates the various properties of the algebra $L_K(E)$ where the graph E has the mentioned property.

Theorem 4.2 *Let E be a finite graph and let K be any field. Then the following are equivalent for the Leavitt path algebra $L = L_K(E)$:*

- (i) *No two distinct cycles in E have a common vertex;*
- (ii) *Every simple left L -module is finitely presented;*
- (iii) *L has finite Gelfand-Kirillov dimension;*
- (iv) *L is the union of a finite ascending chain of graded ideals*

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L \quad (*)$$

with $H_j = I_j \cap E^0$, where $I_0 = \text{Soc}(L)$ and, for each $j \geq 0$, identifying, L/I_j with $L_K(E \setminus H_j)$, I_{j+1}/I_j is the ideal generated by the vertices in all the cycles without exits in $E \setminus H_j$ and $\text{Soc}(L/I_j) = 0$.

- (v) *L is the union of a finite ascending chain of graded ideals*

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L$$

where I_0 is a direct sum of finitely many matrix rings of the form $M_n(K)$ where $n \in \mathbb{N} \cup \{\infty\}$ and for each $j > 1$, I_j/I_{j-1} is a direct sum of finitely many matrix rings of the form $M_n(K[x, x^{-1}])$ where $n \in \mathbb{N} \cup \{\infty\}$.

- (vi) *E^0 is the union of a finite ascending chain of hereditary saturated subsets*

$$H_0 \subset \cdots \subset H_m = E^0$$

where H_0 is the hereditary saturated closure of all the line points in E and, for each $j \geq 0$, $E \setminus H_j$ has no line points and $H_{j+1} \setminus H_j$ is the hereditary saturated closure of the set of vertices in all the cycles without exits in the graph $E \setminus H_j$.

Proof. The equivalence of (i) and (ii) was proved in [10] and that of (i) and (iii) was proved in [5].

Now (i) \Rightarrow (iv) follows from the proof of Theorem 1 of [6]. We give a slightly different streamlined proof. Let $I_0 = \text{Soc}(L)$, so I_0 is the ideal generated by all the line points in E ([12]). Now, for any graded ideal J containing $\text{Soc}(L)$, $\text{Soc}(L/J) = 0$. This is because, as the hereditary saturated set $J \cap E^0 = H$ contains all the line points in E , the finiteness of E implies that the quotient graph $E \setminus H$ contains no sinks and hence no line points. Suppose for $n \geq 0$ we have defined the graded ideal $I_n \supseteq I_0$. Let $H_n = I_n \cap E^0$. Then $E \setminus H_n$ satisfies the same hypothesis as E and has no sinks, so that every vertex in it connects to a cycle. Moreover, we claim that $E \setminus H_n$ contains cycles without exits. To see this, for any given two cycles c, c' in $E \setminus H_n$, define $c \geq c'$ if there is a path from a vertex in c to a vertex in c' . Since no two cycles in $E \setminus H_n$ have a common vertex, \geq is antisymmetric and hence a partial order. Clearly every cycle which is minimal in this partial order has no exits in $E \setminus H_n$. Now $L/I_n \cong L_K(E \setminus H_n)$ and $\text{Soc}(L/I_n) = 0$. Define I_{n+1}/I_n to be the ideal generated by the vertices in all the cycles without exits in $E \setminus H_n$. It is clear that I_{n+1} is a graded ideal of L . By induction on n , after a finite number of steps, we then obtain the chain (*) with the desired properties.

(iv) \Rightarrow (v) By Theorem 5.6 of [12], I_0 is a direct sum of finitely many matrix rings of the form $M_n(K)$ where $n \in \mathbb{N} \cup \{\infty\}$ and, for each $j \geq 0$, I_{j+1}/I_j is,

by Proposition 3.7 of [4], a direct sum of finitely many matrix rings of the form $M_n(K[x, x^{-1}])$ where $n \in \mathbb{N} \cup \{\infty\}$.

(v) \Rightarrow (vi). Obvious from the proof of (iii) \Rightarrow (iv).

Assume (vi). Suppose, by way of contradiction, there is a vertex w which is the base of two distinct cycles g, h . Now $w \notin H_0$ since otherwise, by Lemma 4.1, w will be a line point in E , a contradiction. Let $t \geq 0$ be the smallest integer such that $w \notin H_t$, so $w \in H_{t+1} \setminus H_t$. Now $H_{t+1} \setminus H_t$ is the saturated closure of the set S_t of all the vertices on cycles without exits in the quotient graph $E \setminus H_t$. Then, by Lemma 4.1, $w \in S_t$, a contradiction. This proves (i). ■

Following the ideas in [10], we illustrate Theorem 4.2 by the simplest example of the Toeplitz algebra.

Example 4.3 Let E be the graph with two vertices v, w , an edge f with $s(f) = v, r(f) = w$ and a loop c with $s(c) = v = r(c)$. Since w is the only line point, the socle of $L = L_K(E)$ is $S = \langle w \rangle$ (see [12]) and there is an epimorphism $L \rightarrow K[x, x^{-1}]$ with kernel S mapping v to 1, c to x and c^* to x^{-1} . Thus $L/S \cong K[x, x^{-1}]$ and, moreover, $S = Lw \oplus \bigoplus_{n=0}^{\infty} Lwf^*(c^*)^n$ is the direct sum of simple left ideals in L . We wish to show that every simple left L -module $A = L/M$ is cyclically (hence finitely) presented, where M is a maximal left ideal of L .

If $S \not\subseteq M$, then we have a direct decomposition $S = (S \cap M) \oplus T$ and $L = M \oplus T$. If $1 = \epsilon + \epsilon'$ with ϵ in M and $\epsilon' \in T$, then $M = L\epsilon$ is cyclic.

Suppose $S \subseteq M$. Then there is an irreducible polynomial $p(x) = 1 + k_1x + \dots + k_mx^m \in K[x, x^{-1}]$ such that $M/S = \langle p(x) \rangle$. So $M = Lp(c) + S = Lp(c) + Lw + \bigoplus_{n=0}^{\infty} Lwf^*(c^*)^n = Lp(c) + \bigoplus_{n=0}^{\infty} Lf^*(c^*)^n$ as $wp(c) = w$. Let $N = \bigoplus_{i=0}^{m-1} Lf^*(c^*)^i$. Suppose $r \geq m-1$ and that $f^*(c^*)^t \in Lp(c) + N$ for all $t \leq r$. Then, $f^*(c^*)^{r+1} = f^*(c^*)^{r+1}p(c) - k_1f^*(c^*)^r - \dots - k_mf^*(c^*)^{r+1-m} \in Lp(c) + N$. Thus we conclude that $Lp(c) + S = Lp(c) + N$. Observing that $\{f^*(c^*)^i : i = 0, \dots, m-1\}$ is a set of mutually orthogonal elements, we get $N = \bigoplus_{i=0}^{m-1} Lf^*(c^*)^i = Lb$ where $b = f^* + f^*c^* + \dots + f^*(c^*)^{m-1}$. Further, $p(c)f^*(c^*)^i = 0$ for all i and that $p(c) = p(c)v \in Lv$. Consequently, $M = Lp(c) + S = L(v+b)$ is cyclic. This proves that the simple module L/M is cyclically presented.

REMARK: Observe that, in our proof above, we never used the fact that the polynomial $p(x)$ is irreducible. Since $K[x, x^{-1}]$ is a principal ideal domain, the same argument shows that every left ideal $A \not\subseteq S$ in L is a principal left ideal. Also, if A is a left ideal such that $S \not\subseteq A$ and $A \not\subseteq S$, then decomposing $S = (S \cap A) \oplus T$, we see that the left ideal $A + S = A \oplus T \not\subseteq S$. Thus $A \oplus T$ and hence A is a principal left ideal in L . On the other hand, if $A \subseteq S$, A need not be a principal left ideal. This is clear if $A = S$, as S is a direct sum of infinitely many simple left ideals. In particular, S is not a direct summand of L . Thus we obtain an easy proof of the following proposition which occurs as Theorem 2 in [6].

Proposition 4.4 *Let E be a graph with two vertices v, w and two edges c, f with $s(c) = v = r(c), s(f) = v, r(f) = w$. If $S = \langle w \rangle$ is the two-sided ideal generated by w , then S cannot be a direct summand of $L = L_K(E)$ as a left L -module.*

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